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Some Applications of the Theory of Blocks of Characters of Finite Groups. I*

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I. INTRODUCTION

If G is a finite group and p a fixed prime number, the irreducible representations of G are distributed into disjoint systems, the p-blocks, which are related closely to the arithmetic structure of the group algebra of G. These blocks have been investigated in two preceding papers, Brauer [*1*, *2*]. Different proofs of some of the results have been given by Osima [*10*], Rosenberg [*11*], Iizuka [*7*], Nagao [*9*]; see also Curtis-Reiner [*6*]. It is our aim to give some applications of the theory. In the present first part we are concerned mainly with the problem of obtaining properties of the characters of G from group theoretical information concerning G. In the later parts, we shall use the results for a study of groups of even order; the prime p will be taken as 2.

In Section II of the present paper, we list some of the known results which are used later on.

In Section III, we obtain a number of results concerning the principal p-block. As an application, we introduce certain types of groups, the groups of a given deficiency class for the prime p. These types can be defined inductively by group theoretical properties. On the other hand, they can be characterized by properties of their characters, Section IV. In Section V, the notion of a basic set for a block is introduced and it is shown how basic sets can be used for a discussion of properties of characters.

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II. NOTATION. PRESUPPOSED RESULTS

1. In the following, G will always be a group of finite order, and p will be a fixed prime number. If M is a subset of G, we write $|M|$ for the number of elements of M. The centralizer of M in G will be denoted by $\mathfrak{C}_G(M)$ and the normalizer of M by $\mathfrak{N}_G(M)$.

By the p-regular core $\mathfrak{K}_p(G)$ of G, we mean the maximal normal subgroup of G of an order prime to p (often denoted by $\mathfrak{O}_{p'}(G)$).

2. Let $\bar{\mathbf{Q}}$ denote the algebraic closure of the field $\mathbf{Q}$ of rational numbers. Choose a fixed exponential valuation ν_p of $\bar{\mathbf{Q}}$ which extends the p-adic valuation of $\mathbf{Q}$; $\nu_p(p) = 1$. Let $\mathfrak{o}$ denote the local ring of ν_p in $\bar{\mathbf{Q}}$. Let $\mathfrak{p}$ denote the corresponding prime ideal and set $\Omega = \mathfrak{o}/\mathfrak{p}$. The residue class map of $\mathfrak{o}$ onto Ω will be denoted by an asterisk; $\alpha \to \alpha^*$.

By a representation X of G or, more clearly, by an ordinary representation of G, we shall mean a representation of G in $\bar{\mathbf{Q}}$. We use the term modular representation of G for the representations of G in Ω. An analogous terminology will be used in the case of characters. We can speak of the modular irreducible constituents of an ordinary representation. cf. Appendix of [*1*].

The group algebra of G over Ω will be denoted by $\Gamma(G)$ and its center by $Z(G)$. If M is a subset of G, we write $[M]$ for the element of $\Gamma(G)$ defined by

$$[M] = \sum_{m \in M} m. \tag{2.1}$$

In particular, if $K_1, K_2, \cdots, K_k$ are the conjugate classes of G, the elements $[K_1], [K_2], \cdots, [K_k]$ form a basis of $Z(G)$. For this reason, we refer to $Z(G)$ as the class algebra of G (over Ω). The k irreducible characters of G will be denoted by $\chi_0 = 1, \chi_1, \cdots, \chi_{k-1}$.

3. It will not be possible to list all the known results on blocks which will be used below. We shall state here some significant facts and refer for details to the papers quoted in Section I.

A p-block B of G is a set of irreducible representations of G. Here, we place two irreducible representations X and Y of G in the same block, if there exists a chain

$$X^{(0)} = X, X^{(1)}, \cdots, X^{(r)} = Y \tag{2.2}$$

of irreducible representations $X^{(i)}$ of G such that any two neighbors in (2.2) have a modular irreducible constituent in common. By the modular irreducible representations of the block B, we shall mean the modular irreducible constituents of the representations $X \in B$. Finally, we replace representations by characters and look upon a block B as a set of characters. It will be convenient to use the notation $s(B)$ for the set of indices i for which $\chi_i \in B$.

If F is a modular irreducible representation of G and hence of $\Gamma(G)$, the elements $x \in Z(G)$ are represented by scalar multiples $\psi(z) I$ of the identity; $\psi \in \Omega$. Here, ψ is a linear character of $Z(G)$, i.e. an algebra-homomorphism of $Z(G)$ onto Ω. Actually, ψ depends only on the p-block B to which F belongs. If we write ψ_B for ψ, then $B \to \psi_B$ is a one-to-one correspondence between the set of p-blocks of G and the set of linear characters of the class algebra $Z(G)$. If σ_j is a representative for the conjugate class K_j, we have $\chi_i \in B$, if and only if

$$\psi_B([K_j]) = (| K_j | \chi_i(\sigma_j)/\chi_i(1))^* \tag{2.3}$$

for all j. (As explained above, the asterisk indicates the residue class map $\mathfrak{o} \to \Omega$).

The principal p-block $B_0 = B_0(G)$ of G is the p-block containing the principal character $\chi_0 = 1$. It follows from (2.3) that

$$\psi_{B_0}([K_j]) = (| K_j |)^*. \tag{2.4}$$

4. Let H be a subgroup of G and let λ be a linear function on $Z(H)$ with values in Ω. We denote by λ^G the unique linear function on $Z(G)$ for which

$$\lambda^G([K_j]) = \lambda([K_j \cap H]) \tag{2.5}$$

for $j = 1, 2, \cdots, k$. In particular, if b is a p-block of H and ψ_b the corresponding linear character of $Z(H)$, we may take $\lambda = \psi_b$. Then $\psi_b{}^G$ may or may not be a linear character of $Z(G)$. If we have the former case and if $\psi_b{}^G = \psi_B$, we say that b^G is defined and we set $b^G = B$.

Each p-block B of G determines a class of conjugate p-subgroups of G, the defect groups of B. Their order is p^d where d is the defect of B which can be determined, if we know the degrees $\chi_i(1)$ of the characters $\chi_i \in B$. If the block b of H has the defect group D_0 (in H), then b^G is always defined when $\mathfrak{C}_G(D_0) \subseteq H$. If the block B of G has defect group D, there exist blocks b of $DC_G(D)$ such that $b^G = B$. Each such b has necessarily defect group D.

5. Let π be a fixed p-element of G, say, of order p^α. If v is a p-regular element of $C_G(\pi)$, we have formulas

$$\chi_i(\pi v) = \sum_\rho d_{i\rho}{}^\pi \varphi_\rho{}^\pi(v) \tag{2.6}$$

for $\chi_i \in B$. Here the $\varphi_\rho{}^\pi$ range over the modular irreducible characters of the blocks b of $\mathfrak{C}_G(\pi)$ for which $b^G = B$. The decomposition numbers $d_{i\rho}{}^\pi$ are algebraic integers of the field of the p^α-th roots of unity which do not depend on v. If $c_{\rho\sigma}{}^\pi$ is the Cartan invariant of $\mathfrak{C}_G(\pi)$ belonging $\varphi_\rho{}^\pi$ and $\varphi_\sigma{}^\pi$, then

$$\sum_{i \in s(B)} d_{i\rho}{}^\pi \bar{d}_{i\sigma}{}^\pi = c_{\rho\sigma}{}^\pi. \tag{2.7}$$

On the other hand, if π and π' are two p-elements of G which are not conjugate in G,

$$\sum_{i \in s(B)} d_{i\rho}{}^{\pi} \bar{d}_{i\sigma}{}^{\pi'} = 0. \tag{2.8}$$

By the kernel of a character, we shall always mean the kernel of the corresponding representation. Since π lies in the center of $\mathfrak{C}_G(\pi)$, the kernel of the modular irreducible characters $\varphi_i{}^{\rho}$ of $\mathfrak{C}_G(\pi)$ contains $\{\pi\}$. Hence the $\varphi_i \in b$ can be considered as modular irreducible characters of $\mathfrak{C}_G(\pi)/\{\pi\}$. They form the modular irreducible characters of a block $\tilde{b}$ of $\mathfrak{C}_G(\pi)/\{\pi\}$. If $\tilde{b}$ has defect d_0, b has defect $d_0 + \alpha$. If the Cartan invariant of $\mathfrak{C}_G(\pi)/\{\pi\}$ corresponding to $\varphi_\rho{}^\pi$, $\varphi_\sigma{}^\pi$ is $\tilde{c}_{\rho\sigma}$, then $c_{\rho\sigma}{}^\pi$ in (2.7) is given by $c_{\rho\sigma}{}^\pi = p^\alpha \tilde{c}_{\rho\sigma}$.

6. If Y is a representation of G and τ an automorphism of G, a representation Y^τ of G is defined by $g \to Y(\tau^{-1}(g))$ for all $g \in G$. An analogous remark applies to modular representations. It is clear that τ maps each p-block B on a p-block B^τ.

In particular, if G is a normal subgroup of a group T, we may take τ as the automorphism $g \to t^{-1}gt$ of G produced by a fixed element t of T. We write here Y^t for Y^τ, B^t for B^τ, and we say that Y and Y^t are representations of G associated in T, and that B and B^t are p-blocks of G associated in T.

III. Properties of the Principal Block

We need a simple lemma.

Lemma 1. *Let H be a normal subgroup of G. For each p-block B of G, there exists a family $\{b^g\}$ of p-blocks of H associated in G with the following property: If W is an ordinary or modular irreducible representation in B, the irreducible constituents of $W \mid H$ lie in blocks b^g.*

Proof. It follows from Clifford's theorem that for each W, the irreducible constituents of $W \mid H$ are associated in G and hence lie in a family of associated blocks of H. If we apply this to an ordinary irreducible representation $X \in B$ and to a modular irreducible constituent F of X, we obtain both times the same family of blocks of H. It then follows for all X_i in a chain (2.2) that the irreducible constituents of $X_i \mid H$ lie in a family of associated blocks of H and this implies the lemma.

As already mentioned, the principal p-block $B_0 = B_0(G)$ of G is the p-block containing $\chi_0 = 1$. We prove

Theorem 1. *The intersection of the kernels of the ordinary irreducible representations $X_i \in B_0$ is the p-regular core $\mathfrak{K}_p(G)$.*

Proof. Since $\chi_0 \mid \mathfrak{R}_p(G)$ is the principal character of $\mathfrak{R}_p(G)$, it follows from Lemma 1 with $H = \mathfrak{R}_p(G)$ that for $\chi_i \in B_0$ all irreducible constituents of $\chi_i \mid \mathfrak{R}_p(G)$ lie in the principal p-block b_0 of $\mathfrak{R}_p(G)$. Since $\mathfrak{R}_p(G)$ is p-regular, b_0 consists only of one ordinary character, the principal character. If X_i is the representation with the character χ_i, it follows that $X_i \mid \mathfrak{R}(G)$ has the principal representation of $\mathfrak{R}_p(G)$ as its only irreducible constituent. This implies that $\mathfrak{R}_p(G)$ belongs to the kernel of $X_i \in B_0$.

Let H denote the intersection of the kernels of the representations X_i. As shown, $H \supseteq \mathfrak{R}_p(G)$. We may consider the ordinary and modular representations in $B_0(G)$ as representations of G/H and then the principal blocks $B_0(G)$ and $B_0(G/H)$ coincide. In particular, both blocks have the same decomposition numbers and Cartan invariants. This implies that both have the same defect (for instance by [*1*, (6C)]). Since the defect of the principal block is the exponent of the highest power of p which divides the group order, we find $\nu_p(\mid G \mid) = \nu_p(\mid G/H \mid)$. It follows that H is p-regular and this implies $H = \mathfrak{R}_p(G)$.

LEMMA 2. *If the irreducible character χ_i belongs to B_0, each algebraically conjugate character χ_i' belongs to B_0.*

This follows from (2.3), (2.4), if we use the fact that the value of χ_i' for the elements of a class K_j is equal to the value of χ_i for the elements of a class $K_{j'}$, with $\mid K_j \mid = \mid K_{j'} \mid$.

COROLLARY 1. *Set*

$$N = \sum_{i \in s(B_0)} \chi_i(1).$$

Then $\mid G : \mathfrak{R}_p(G) \mid$ *lies between N and an upper bound depending only on N.*

Proof. It follows from Theorem 1 that

$$\sum_{i \in s(B_0)} \chi_i(1)^2 \leq \mid G : \mathfrak{R}_p(G) \mid$$

and this implies $N \leqslant \mid G : \mathfrak{R}_p(G) \mid$. On the other hand, by Lemma 2,

$$\theta = \sum_{i \in s(B_0)} \chi_i$$

is a rational-valued character of G which by Theorem 1 belongs to a faithful representation of $G/\mathfrak{R}_p(G)$. Now a theorem of Schur [*11*] gives the required upper estimate of $\mid G : \mathfrak{R}_p(G) \mid$.

We can give an analogue for modular representations. We have

THEOREM 2. *The intersection L of the kernels of the modular irreducible* *representations F_ρ in $B_0(G)$ is the group $\mathfrak{O}_{p'p}(G)$, i.e. the maximal normal subgroup $L_1 \supseteq \mathfrak{K}_p(G)$ of G for which $L_1/\mathfrak{K}_p(G)$ is a p-group.*

Proof. Clearly, $L \supseteq \mathfrak{K}_p(G)$. For each p-regular element v of L and for each modular irreducible character, $\varphi_\rho \in B_0$, we have $\varphi_\rho(v) = \varphi_\rho(1)$. It then follows from (2.6) with $\pi = 1$ that $\chi_i(v) = \chi_i(1)$ for each $\chi_i \in B_0$. Hence $v \in \mathfrak{K}_p(G)$. This shows that $L/\mathfrak{K}_p(G)$ is a p-group. Hence $L \subseteq \mathfrak{O}_{p'p}(G)$. On the other hand, if we consider $\varphi_\rho \in B_0$ as modular irreducible character of $G/\mathfrak{K}_p(G)$, its kernel includes the normal p-subgroup $\mathfrak{O}_{p'p}(G)/K_p(G)$ (cf. [*1*], (9D)). This implies $\mathfrak{O}_{p'p}(G) \subseteq L$ and we must have equality, q.e.d.

We can now prove the following analogue of Corollary 1.

Corollary 2. *Let N_0 be the sum of the degrees of the nonprincipal irreducible characters $\varphi_\rho \in B_0$. Then $G/\mathfrak{O}_{p'p}(G)$ is isomorphic with a subgroup of $GL(N_0, p)$.*

Indeed, if we form the modular representation F of degree N_0, which splits completely into the different representations with the characters $\varphi_\rho \in B_0$, $\varphi_\rho \neq 1$, then tr $F(g)$ lies in the prime field $\Lambda \subset \Omega$. It follows that F is equivalent with a representation in Λ. By Theorem 2, F has the kernel $\mathfrak{O}_{p'p}(G)$ and the corollary becomes evident.

Since we have $\mathfrak{O}_{p'p}(G) = G$, if and only if G has a normal p-complement, Theorem 2 also implies the following known result ([*5*], §29).

Corollary 3. *A group G has a normal p-complement, if and only if the modular principal character $\varphi_0 = 1$ is the only modular irreducible character in $B_0(G)$.*

In the case of the principal block, the formulas (2.6) take a somewhat simpler form. This is a consequence of the following result.

Theorem 3. *Let H be a subgroup of G, let b be a block of H with the defect group D (in H), and assume that $\mathfrak{C}_G(D) \subseteq H$. Then $b^G = B_0(G)$, if and only if b is the principal block $B_0(H)$ of H.*

Proof.[1] (a) Set $B = B_0(H)^G$, $B_0 = B_0(G)$. Using the definition of the block B, we see from (2.4) for $B_0(H)$ and (2.5) that

$$\psi_B(K_j) = |K_j \cap H|^*, \qquad (j = 1, 2, \cdots, k). \tag{3.1}$$

Partition K_j into subclasses L_μ of elements conjugate with regard to D. Then $|L_\mu|$ is a power of p and $|L_\mu| = 1$, if and only if the only element of L_μ lies in $\mathfrak{C}_G(D) \subseteq H$. It follows that the complement of $K_j \cap H$ in K_j is a union

[1] A somewhat different proof will be presented elsewhere in a more general setting.

of subclasses L_μ with $|L_\mu| > 1$. This implies $|K_j| \equiv |K_j \cap H|$ (mod p) and (3.1) and (2.4) for $B_0 = B_0(G)$ yield

$$\psi_B(K_j) = \psi_{B_0}(K_j)$$

for all j. It follows that $B_0(H)^G = B = B_0(G)$.

(b) Conversely, assume that $b^G = B_0 = B_0(G)$. If it is not true that $b = B_0(H)$, choose a counterexample in which $\nu_p(G:D)$ is minimal and among the counterexamples which satisfy this condition, choose one in which $|G|$ is minimal.

According to Section II, there exist blocks $\tilde{b}$ of $D\mathfrak{C}_H(D) = D\mathfrak{C}_G(D)$ with the defect group D for which $\tilde{b}^H = b$. Then $\tilde{b}^G = b^G = B_0$. Here, $\tilde{b}$ cannot be the principal block of $D\mathfrak{C}_H(D)$, since otherwise by part (a), we would have $b = B_0(H)$. Changing the notation, we may as well assume

$$H = DC_G(D)$$

replacing b by $\tilde{b}$.

Then b contains a unique irreducible character θ whose kernel includes D and which as character of $D\mathfrak{C}_G(D)/D = H/D$ has defect 0.

Suppose first that there do not exist p-elements $x \in \mathfrak{N}_G(D)$, $x \notin H$ for which $\theta^x = \theta$. Then in the sense of [*1*], (12A), θ is associated with a p-block B of G with the defect group D and, by [*2*, (2D)], $B = b^G = B_0$. But in the sense of [*1*, (12A)], the principal block B_0 of G is associated only with the principal character of $D\mathfrak{C}_G(D)$. Hence θ is this principal character. This is impossible, since $\theta \in b$ and $b \neq B_0(D\mathfrak{C}_G(D))$.

It follows that there exist p-elements $x \in \mathfrak{N}_G(D)$ such that $x \notin D\mathfrak{C}_G(D)$ and that $\theta^x = \theta$. We may assume without restriction that

$$x^p \in D\mathfrak{C}_G(D) \lhd \mathfrak{N}_G(D).$$

Then $R = \{D\mathfrak{C}_G(D), x\}$ is a group containing $D\mathfrak{C}_G(D)$ as normal subgroup of index p. Set $b_1 = b^R$. Then $b_1{}^G = b^G = B_0$. Since b_1 has larger defect than b (cf. [*2*], (2F)), it follows from the minimal choice of our counterexample that $b_1 = B_0(R)$. Since then $b^R = B_0(R)$ it follows in the same manner that $R = G$. Thus,

$$D \lhd G; \qquad H = D\mathfrak{C}_G(D) \lhd G; \qquad |G:H| = p.$$

Since $\theta^x = \theta$, we can extend θ to an irreducible character χ of G; $\theta(1) = \chi(1)$. Let B be the p-block of G containing χ and let D_1 be a defect group of B. Since $D \lhd G$, we have $D \subseteq D_1$, (cf. [*1*, (9F)]) and hence $\mathfrak{C}_G(D_1) \subseteq \mathfrak{C}_G(D) \subseteq H$. We shall show that $B = B_0$ by showing that $\psi_B = \psi_{B_0}$ Since $\chi \in B$, by (2.3)

$$\psi_B(K_j) = (|K_j|\chi(\sigma_j)/\chi(1))^*. \tag{3.2}$$

On the other hand since $B_0 = b^G$ and $\theta \in b$, by (2.3) for b and by (2.5)

$$\psi_{B_0}(K_j) = \Big(\sum_{\mu} | k_{j\mu} | \theta(\sigma_{j\mu})/\theta(1)\Big)^* \tag{3.3}$$

where $k_{j\mu}$ ranges over all conjugate classes of H contained in K_j and where $\sigma_{j\mu} \in k_{j\mu}$. Since χ extends θ, we may replace θ by χ. We have $\chi(\sigma_{j\mu}) = \chi(\sigma_j)$. If K_j meets H, then K_j is the disjoint union of the $k_{j\mu}$, $| K_j | = \sum_\mu | k_{j\mu} |$ and comparison of (3.2) with (3.3) yields $\psi_{B_0}(K_j) = \psi_B(K_j)$. If $K_j \cap H = \emptyset$, then by (3.3) $\psi_{B_0}(K_j) = 0$. Suppose that $\psi_B(K_j) \neq 0$. Then ([*l*], (8A)) there exist elements $\sigma \in K_j$ such that the defect group D_1 belongs to $\mathfrak{C}_G(\sigma)$. This implies

$$\sigma \in \mathfrak{C}_G(D_1) \subseteq \mathfrak{C}_G(D) \subseteq H$$

and we have $\sigma \in K_j \cap H$, a contradiction.

We now have $\psi_B = \psi_{B_0}$ and hence $B = B_0$. Apply now Lemma 1 to B_0. Since $\chi_0 \in B_0$, $\chi_0 | H \in B_0(H)$, it follows that all irreducible constituents of $\chi | H$ lie in $B_0(H)$. Since $\chi | H = \theta \in b \neq B_0(H)$, this is a contradiction and Theorem 3 is proved.

If H is the centralizer of a p-element π as in (2.6), then π belongs to the defect group D of every p-block b of H. Hence $\mathfrak{C}_G(D) \subseteq \mathfrak{C}_G(\pi) = H$, and, by Theorem 3, $b^G = B_0(G)$, if and only if $b = B_0(H)$. Hence we have

Corollary 4. *If $B = B_0(G)$, then $\varphi_\rho{}^\pi$ in (2.6) ranges over the modular irreducible characters of $B_0(\mathfrak{C}_G(\pi))$.*

For $B \neq B_0(G)$, we may have several blocks b of H with $b^G = B$. They need not even have the same defect.

On combining Corollaries 3 and 4, we obtain

Corollary 5. *Suppose that π is a p-element of G for which $\mathfrak{C}_G(\pi)$ has a normal p-complement. Then for every p-regular element v of $\mathfrak{C}_G(\pi)$ and for every $\chi_i \in B_0(G)$,*

$$\chi_i(\pi v) = \chi_i(\pi).$$

The following remark of a different nature is sometimes useful.

Remark. The characters χ_i of degree 1 in $B_0(G)$ form a group under multiplication and hence the number of such characters χ_i is a divisor of the index of the commutator subgroup G' of G.

This is an immediate consequence of (2.3) and (2.4).

IV. Groups of a Given Deficiency Class

We first prove a lemma.

Lemma 3. *Let B be a block of G of defect d with the defect group D. If π is an element of the center $\mathfrak{Z}(D)$, there exist blocks b of $\mathfrak{C}_G(\pi)$ of defect d such that $b^G = B$.*

Proof. Set $a = \nu_p(|G|)$. There exist characters $\chi_i \in B$ with

$$\nu_p(\chi_i(1)) = a - d.$$

By the definition of the defect group in [1], we can find a p-regular element v of G such that D is a p-Sylow group of $\mathfrak{C}_G(v)$ and that $\psi_B([K]) \neq 0$ for the conjugate class K of v. It follows from (2.3) that $\chi_i(v) \not\equiv 0 \pmod{\mathfrak{p}}$. Since $v \in \mathfrak{C}_G(\pi)$, this implies $\chi_i(\pi v) \not\equiv 0 \pmod{\mathfrak{p}}$. Now (2.6) shows that there exist a block b of $\mathfrak{C}_G(\pi)$ with $b^G = B$ and a modular irreducible character $\varphi_\rho{}^\pi$ in b such that

$$\varphi_\rho{}^\pi(v) \not\equiv 0, \qquad d_{i\rho}{}^\pi \not\equiv 0 \pmod{\mathfrak{p}}. \tag{4.1}$$

Since the modular characters of a block can be expressed by the ordinary characters of the block (restricted to p-regular elements) with integral coefficients, it follows that there exists an ordinary irreducible character $\xi \in b$ with $\xi(v) \not\equiv 0 \pmod{\mathfrak{p}}$.

Let L be the conjugate class of $\mathfrak{C}_G(\pi)$ which contains v. Since $|L| \xi(v)/\xi(1)$ is an algebraic integer, we find

$$\nu_p(\xi(1)) \leqslant \nu_p(|L|) = \nu_p(|\mathfrak{C}_G(\pi)|) - \nu_p(|\mathfrak{C}_G(\pi v)|).$$

Now, D is a p-Sylow group of $\mathfrak{C}_G(v)$ and also of $\mathfrak{C}_G(\pi v)$. Thus,

$$\nu_p(|\mathfrak{C}_G(\pi v)|) = d$$

and we find

$$\nu_p(\xi(1)) \leqslant \nu_p(|\mathfrak{C}_G(\pi)|) - d.$$

This shows that the block b of ξ has at least defect d. Since the defect of b cannot be larger than the defect d of $b^G = B$, the lemma is proved.

We now define types of finite groups using characters.

Definition. Let r be an integer. A finite group G will be called of *deficiency class* r for the prime p, if *there does not exist a nonprincipal block of defect* $d \geqslant r$.

The following Remarks 1 and 2 are obvious. Remark 3 follows from [1], (9F), and Remark 4 from [2], (2G).

REMARKS. 1. If p^r does not divide the order $|G|$, G is of deficiency class r.

2. Every p-group has deficiency class 0 for p.

3. If Q of order p^n is a normal p-subgroup of G and if G is of deficiency class $r \leqslant n$, then G is of deficiency class 0.

4. If Q of order p^n is a p-subgroup of the center of G, then G is of deficiency class r, if and only if G/Q is of deficiency class $r - n$. The same fact still holds if Q of order p^n is a subgroup of G for which $G = Q\mathfrak{C}_G(Q)$.

The reason for our interest in the deficiency classes lies in the fact that for $r > 0$ we can characterize them inductively by properties of the abstract group. We prove

THEOREM 4. *If G is of deficiency class r for the prime p then for every p-element π of G, $\mathfrak{C}_G(\pi)$ is of deficiency class r. Conversely, if for every element π of order p the group $\mathfrak{C}_G(\pi)$ is of deficiency class r and if r is positive, then G is of deficiency class r.*

Proof. Let π be a p-element. If $\mathfrak{C}_G(\pi)$ has a non-principal p-block b of defect d, then $b^G = B$ is defined and has defect at least d. By Theorem 3, $B \neq B_0(G)$. If G has deficiency class r, then $d < r$. It follows that $\mathfrak{C}_G(\pi)$ has deficiency class r.

Conversely, assume that G is not of deficiency class r. Let B be a non-principal p-block of defect $d \geqslant r$ and let D be the defect group. Since $r > 0$, we can choose an element of order p in the center of D. Now Lemma 3 shows that $\mathfrak{C}_G(\pi)$ contains a p-block b of defect d with $b^G = B$. Again, b cannot be $B_0(\mathfrak{C}_G(\pi))$ and hence $\mathfrak{C}_G(\pi)$ is not of deficiency class r as we had to show.

In the case of the groups $\mathfrak{C}_G(\pi)$ with $\pi \neq 1$, we can apply Remark 4 above. Hence Theorem 4 furnishes a characterization of the groups of deficiency class $r > 0$. However, we cannot distinguish the groups of deficiency class 0 in an analogous manner among the groups of deficiency class 1.

A group G has deficiency class 0, if $B_0(G)$ is the only p-block. If $\nu_p(|G|) = a$, then $B_0(G)$ contains at most p^{2a} irreducible characters, cf. Brauer-Feit [3]. Hence G has at most class number p^{2a}. It follows that for given a, there are only finitely many groups G, Landau [8]. Thus, we have

THEOREM 5. *There exist only finitely many groups G of deficiency class 0 for p for which the exact power p^a of p in $|G|$ is given.*

For $p = 2$, we can go a bit further.

THEOREM 6. *Consider groups G of even order of deficiency class 1 for $p = 2$ for which the exact power 2^a of 2 in the group order $|G|$ is given. There exist only finitely many such simple groups. Also, there exist only finitely many groups G which have more than one conjugate class consisting of involutions.*

Proof. If x is an element of order p of a group G of deficiency class 1, then by Theorem 4 and Remark 3 above, $\mathfrak{C}_G(x)$ has deficiency class 0. If $\nu_p(|G|) = a$ is given, the order of $\mathfrak{C}_G(x)$ is bounded by Theorem 5. If $p = 2$, the results of Brauer-Fowler *[4]* can be applied and yield the statements.

The C.I.T. groups of Suzuki *[13]* and in particular Suzuki's simple groups have deficiency class 1 for $p = 2$ and a number of other simple groups of this class are known. The Mathieu group M_{24} has deficiency class 0 for $p = 2$.

V. Basic Sets

In the formulas (2.6), the modular irreducible characters of the groups $\mathfrak{C}_G(\pi)$ appear. Since it may be difficult to determine these characters, we shall discuss a modification of the method.

If G is a group and p a prime, we shall denote by G^0 the set of p-regular elements of G and by $\chi_i{}^0$ the restriction of the character χ_i to G^0. If B is a fixed p-block of G, the functions $\chi_i{}^0$ with $\chi_i \in B$ generate a module M_B with regard to the ring $\mathbf{Z}$ of integers. By a basic set φ_B for B, we mean any basis $\{\varphi_\rho\}$ of M_B. Thus, we have formulas

$$\chi_i{}^0 = \sum_\rho d_{i\rho}\varphi_\rho \qquad \text{for} \qquad i \in s(B) \tag{5.1}$$

with $d_{i\rho} \in \mathbf{Z}$. Moreover, we set

$$c_{\rho\sigma} = \sum_{i \in s(B)} d_{i\rho} d_{i\sigma} \tag{5.2}$$

for φ_ρ, $\varphi_\sigma \in \varphi_B$.

In particular, the modular irreducible characters in B form a basic set φ_B. If this φ_B is used, the $d_{i\rho}$ in (5.1) are the decomposition numbers and the $c_{\rho\sigma}$ the Cartan invariants. We use the same terms in the case of an arbitrary basic set. It is obvious in which manner the $d_{i\rho}$ and the $c_{\rho\sigma}$ will change, if the basic set φ_B is replaced by another basic set. In particular, the matrix $(c_{\rho\sigma})$ is the matrix of a quadratic form $\underset{\sim}{Q}$. If the basic set is changed, $\underset{\sim}{Q}$ is replaced by an equivalent form. We note

Theorem 7. *Let p and d be given. The quadratic form $\underset{\sim}{Q}$ associated with a p-block of defect d belongs to one of a finite number of classes of positive definite quadratic forms.*

Indeed, it is clear that $\underset{\sim}{Q}$ is nonnegative. If we use the set of modular irreducible characters of B for φ_B, we know that the number of members of φ_B is at most equal to p^{2d} *[3]*, and that the largest elementary divisor of $(c_{\rho\sigma})$ is p^d and, in particular, that det $(c_{\rho\sigma})$ is apower of p (*[1]*, (6C)). This suffices to establish Theorem 7.

COROLLARY 6. *There exist bounds $\gamma(p^d)$ depending only on p^d such that for each p-block of defect d of any finite group, a basic set can be chosen such that the Cartan invariants are at most equal to $\gamma(p^d)$.*

Using the reduction theory of quadratic forms, we can give explicit estimates for $\gamma(p^d)$. Actually, because of the form (5.2) of the $c_{\rho\sigma}$, better estimates can be obtained by direct methods. This will be shown elsewhere.

COROLLARY 7. *If $B = B_0$ is the principal p-block, a basic set φ_B can be chosen such that $1 \in \varphi_B$ and that the Cartan invariants $c_{\rho\sigma}$ lie below a bound $\gamma_0(p^n)$ depending only on the highest power p^n of p which divides $|G|$.*

Proof. We first choose a basic set φ_B in accordance with Corollary 1. Since $\chi_0 = 1$ belongs to $B = B_0$, we have

$$1 = \sum d_{0\rho}\varphi_\rho$$

where φ_ρ ranges over φ_B. It follows from (5.2) with $\rho = \sigma$ that $d_{0\rho}{}^2 \leqslant c_{\rho\rho} \leqslant \gamma(p^n)$. Moreover, if φ_B consists of l_B functions, the l_B numbers $d_{0\rho}$ are relatively prime. Hence we can find an $(l_B \times l_B)$ matrix T with the following properties: (1) In the first row, the l_B coefficients $d_{0\rho}$ appear. (2) All coefficients lie in $\mathbf{Z}$ and their absolute value lies below a bound depending only on p^n. (3) The determinant is ± 1.

Now the linear transformation with the matrix T changes the set φ_B into a basic set with the required properties.

Let π now be a p-element of a given finite group and let b be a p-block of $\mathfrak{C}_G(\pi)$ such that b^G is a given p-block B of G. Select a basic set φ_b of b. It is clear that we can introduce the elements of φ_b instead of the irreducible modular characters in b in (2.6). We then obtain formulas of the same form as (2.6), which we write again in the form

$$\chi_i(\pi v) = \sum_b \sum_\rho d_{i\rho}{}^\pi \varphi_\rho{}^\pi(v), \qquad (\chi_i \in B), \tag{5.3}$$

where v is a p-regular element of $\mathfrak{C}_G(\pi)$, where b ranges over the set of blocks of $\mathfrak{C}_G(\pi)$ which satisfy $b^G = B$, and where $\varphi_\rho{}^\pi$ ranges over the members of a basic set φ_b for b. If π has order p^α, the $d_{i\rho}{}^\pi$ are algebraic integers of the field of the p^α-th roots of unity. Exactly as in (2.7), we have

$$\sum_{i \in s(B)} d_{i\rho}{}^\pi \bar{d}_{i\sigma}{}^\pi = c_{\rho\sigma}{}^\pi \tag{5.4}$$

where $c_{\rho\sigma}{}^\pi$ is now the Cartan invariant belonging to φ_b; $\varphi_\rho{}^\pi, \varphi_\sigma{}^\pi \in \varphi_b$.

Let π' also be a p-element, let b' be a p-block of $\mathfrak{C}_G(\pi')$ with $(b')^G = B$ and let $\varphi_\sigma^{\pi'}$, be a member of a basic set $\varphi_{b'}$. If either $\pi = \pi'$, $b \neq b'$ or if π and π' are not conjugate in G, it follows from (2.7) and (2.8) that

$$\sum_{i \in s(B)} d_{i\rho}^{\pi} \bar{d}_{i\sigma}^{\pi'} = 0 \tag{5.5}$$

We state

THEOREM 8. *Let p and d be given. For all finite groups G, there exist only finitely many "types" of p-blocks B of defect d. For each type, the defect group D of B is determined as abstract group. Furthermore, we have a fixed system of elements of D*

$$\pi_0 = 1, \pi_1, \cdots, \pi_m \tag{5.6}$$

which represent the different conjugate classes of G which meet D. Moreover for each $\pi = \pi_\mu$, $1 \leqslant \mu \leqslant m$, the coefficients $d_{i\rho}^{\pi}$ with $i \in s(B)$ are determined provided that suitable basic sets are used.

This is fairly obvious. For given p and d, we have only finitely many possibilities for D and for the choice of the system (5.6). We have already used that the number k_B of characters $\chi_i \in B$ is at most p^{2d}. For fixed $\pi = \pi_j$, let $l_B^{(j)}$ denote the total number of functions φ_ρ^{π} in basic sets φ_b belonging to blocks b of $\mathfrak{C}_G(\pi)$ with $b^G = B$. If we use for a moment again the irreducible modular characters as basic set, we have ([2], (7D))

$$\sum_{j=0}^{m} l_B^{(j)} = k_B \leqq p^{2d}. \tag{5.7}$$

For each block b with $b^G = B$, the defect d_0 is at most equal to d. Suppose that in each case a basic set φ_b is used which satisfies the condition of Corollary 6. Consider a fixed φ_ρ^{π}. The corresponding $d_{i\rho}^{\pi}$ are algebraic integers of a known algebraic number field. Now (5.4) with $\rho = \sigma$ yields

$$\sum_i | d_{i\rho}^{\pi} |^2 \leqq \gamma(p^{d_0}). \tag{5.8}$$

For each algebraic conjugate $(d_{i\rho}^{\pi})'$ of $d_{i\rho}^{\pi}$, there exists an algebraically conjugate character $\chi_{i'}$ of χ_i which belongs to B and for which

$$d_{i'\rho}^{\pi} = (d_{i\rho}^{\pi})'.$$

This follows easily from (5.3). Now (5.8) shows that we have only finitely many possibilities for each $d_{i\rho}^{\pi}$. Since (5.7) shows that we have at most p^{4d} coefficients $d_{i\rho}^{\pi}$, this implies Theorem 8.

COROLLARY 8. *Let B be a p-block of a group G of given type. Let π be a p-element of G and let v be a p-regular element of $\mathfrak{C}_G(\pi)$. If π is one of the elements (5.6), and if the corresponding basic sets φ_b are known, the values of the characters $\chi_i \in B$ for the element πv are determined. If π is not conjugate in G to an element (5.6), then $\chi_i(\pi v) = 0$ for $\chi_i \in B$.*

The first statement follows from (5.3) while the second is a consequence of [*2*], (7A).

The proof of Theorem 8 does not give a practical way of finding all possible types of p-blocks, even in fairly simple cases. We shall therefore discuss some modifications.

Let B be a p-block of G and let $s(B)$ as before denote the set of indices i for which $\chi_i \in B$. It will be convenient to speak of a column $\mathfrak{z}$ belonging to B if for each $i \in s(B)$ a complex number z_i is given. These columns form a vector space with the usual operations. If $\mathfrak{z} = \{z_i\}$ and $\mathfrak{u} = \{u_i\}$ are two columns, we introduce an inner product by

$$(\mathfrak{z}, \mathfrak{u}) = \sum_{i \in s(B)} z_i \bar{u}_i .$$

If T is a ring, we say that $\mathfrak{z}$ lies in T, if all $z_i \in T$.

In discussing types of p-blocks, we may assume that the defect d is positive, since the case $d = 0$ is trivial, cf. [*1*], (6E). We take the view that the case of a fixed defect group D and a definite system (5.6) of elements of D are considered. For each $\pi = \pi_\mu$, we have to consider all blocks b of $\mathfrak{C}_G(\pi)$ with $b^G = B$. Then the defect group D_0 of b can be taken as a subgroup of D. Actually, the discussion can be reduced to that of blocks $\bar{b}$ of $\mathfrak{C}_G(\pi)/\{\pi\}$ with the defect group $\bar{D} = D_0/\{\pi\}$. If $\pi \neq 1$, i.e., if $\mu > 0$, then $|\bar{D}| < |D|$ and we may assume in the way of induction that we know the Cartan invariants for suitable basic sets of $\bar{b}$ and then of b. Now, as in the proof of Theorem 8, the formulas (5.4) can be used to discuss the possibilities for the columns $\mathfrak{d}_\rho^\pi$ with the coefficients $d_{i\rho}^\pi$, $i \in s(B)$. We do not know in general how many blocks b of $\mathfrak{C}_G(\pi)$ with $b^G = B$ exist. However, (5.7) shows that the total number of columns $\mathfrak{d}_\rho^\pi$ with $\pi \neq 1$ is less than p^{2d}. Of course, the equations (5.5) represent further necessary conditions for these columns.

Suppose that we have made a definite choice for the $d_{i\rho}^\pi$ with $\pi = \pi_1$, π_2, $\cdots$, π_m. It remains to discuss the columns $\mathfrak{d}_\rho^1$ with $\pi = \pi_0 = 1$. Let $\mathfrak{M}$ denote the Z-module consisting of all columns in $\mathbf{Z}$ which are orthogonal to all columns $\mathfrak{d}_\rho^\pi$ with $\pi = \pi_1$, π_2, $\cdots$, π_m. By (5.7), the dimension of $\mathfrak{M}$ is $l_B^{(0)}$. In fact, if we take the modular irreducible characters of B as basic set φ_B, ($\pi = 1$, $b = B$), we see that the corresponding $l_B^{(0)}$ columns $\mathfrak{d}_\rho^1$ form a $\mathbf{Z}$-basis of M. This follows from the known fact (cf. [*1*, §5]) that the greatest common divisor of all minors of degree $l_B^{(0)}$ of these columns is 1.

If we now choose an arbitrary $\mathbf{Z}$-basis $\{\mathfrak{d}_\mu^{(1)}\}$ of M, this means that we decide on a particular choice of the basic set φ_B. The corresponding Cartan invariants can then be obtained from (5.4).

In order to obtain further restricting conditions, we choose a generalized character θ of D and set

$$a_i(\theta) = (\chi_i \mid D, \theta) \tag{5.9}$$

for $i \in s(B)$; we have a column $\mathfrak{a}(\theta)$ in $\mathbf{Z}$. Using (5.3), we can write this in the form

$$| D | \, a_i(\theta) = \sum_{\pi \in D} \chi_i(\pi) \, \bar{\theta}(\pi) = \sum_{\pi} \sum_{b} \sum_{\rho} d_{i\rho}{}^{\pi} \varphi_\rho{}^{\pi}(1) \, \bar{\theta}(\pi). \tag{5.10}$$

If $| D | = p^d$ and if we know the values $\varphi_\rho{}^\pi(1)$, this can be considered as a congruence mod p^d for the columns $\mathfrak{d}_\rho{}^\pi$. We always have at least some information concerning the $\varphi_\rho{}^\pi(1)$. Indeed, if $p^{h(\pi)}$ divides the order of $\mathfrak{C}_G(\pi)$, it follows from [*1*], (3F), that

$$\sum_{\sigma} c_{\rho\sigma}{}^{\pi} \varphi_\sigma{}^{\pi}(1) \equiv 0 \ (\mathrm{mod}\ p^{h(\pi)}) \tag{5.11}$$

where the sum extends over all $\varphi_\sigma{}^\pi \in \varphi_b$.

In our discussion so far, we used the estimate (5.7) for k_B. Actually, it is not difficult to show that if π is an element of the center of D, then there exists a p-block b of $\mathfrak{C}_G(\pi)$ with $b^G = B$ with the following property. For each $\chi_i \in B$, there exists a $\varphi_\rho{}^\pi \in b$ with $d_{i\rho}{}^\pi \neq 0$. This can often be used to obtain much better estimates for k_B. There are a number of further remarks which facilitate the discussion.

We note the following corollary of Theorem 8.

COROLLARY 9. *If B is a p-block of the group of a given type, if π is a p-element of G and if the group $\mathfrak{C}_G(\pi)$ is known, the values of the characters $\chi_i \in B$ for the elements πv are determined where v is a p-regular element of $\mathfrak{C}_G(\pi)$.*

As a final result, we mention

THEOREM 9. *Let G be a finite group, let p be a prime and let*

$$\pi_0 = 1, \pi_1, \cdots, \pi_m$$

be a set of representatives for the conjugate classes of G consisting of p-elements. Assume that, for $i > 0$, the group $\mathfrak{C}_G(\pi_i)$ is known and that we know which of the conjugate classes of these groups are fused in G, i.e., which of them belong to the same conjugate class of G. Then the values of the linear characters $\psi_B([K])$ for p-singular classes can be determined. (However, we are not able to say how *many blocks B of defect 0 occur). For each block b of $\mathfrak{C}_G(\pi_i)$ with $i > 0$, we can determine which ψ_B corresponds to b^G.*

If we know the $d_{i\rho}{}^{\pi}$ for $\pi = \pi_\mu$ with $\mu > 0$ as assumed in Theorem 8, we can then find the values of $\chi_i \in B$ for p-singular elements. Then (5.10) with $\theta = 1$ can be used to find the value of $\chi_i(1)$ (mod p^n) where p^n is the highest power of p dividing $|G|$. In a similar manner, we obtain congruences for $\chi_i(v)$ where v is a p-regular element for which $|\mathfrak{C}_G(v)|$ is divisible by p. Finally, it follows from (5.5) that

$$\sum_{i \in s(B)} \chi_i(w)\, d_{i\rho}{}^{\pi} = 0 \tag{5.12}$$

where w is a p-regular element of G and $\pi = \pi_\mu$ with $\mu > 0$.

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